COMPACTIFICATIONS OF SYMMETRIC VARIETIES

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TOHRU UZAWA

ABSTRACT. We construct canonical compactifications of symmetric varieties over the ring of integers.

CONTENTS

1. INTRODUCTION

This is an expanded version of the talk given at the July 1996 meeting on representation theory at the Research Institute of Mathematical Sciences, Kyoto.

In the talk, the following topics were covered.

- The general theory of compactifications over an algebraically closed field of characteristic not equal to 2.
- Construction of a compactification over Z, the ring of integers, for the case PGL_3/PO_3.

During the write up of the talk, the following generalizations have been obtained.

- Treatment of involutions over fields of characteristic 2. Especially the existence of σ-split tori for non-trivial involutions.
- Construction of compactification over the ring of integers for symmetric varieties of inner type.

These have also been included in this note. Please note that a complete write up, with applications to representation theory, is now in preparation.

Let us now proceed with the formal introduction.

Let k be a field and let G be a reductive group scheme defined over k. We denote by σ an involution of G defined over k and by $H = G^\sigma$ the fixed point subgroup of σ. Recall that a symmetric variety is by definition the affine variety G/H defined over k. The purpose of this note is two-fold: to clarify the structure of equivariant compactifications of symmetric varieties by using a method indicated in [26]; on the way its connection with the theory of Luna-Vust, Brion, Pauer, and Knop will be clarified. The method here is to reduce the classification of compactifications of symmetric varieties to Weyl group invariant compactifications of a certain torus associated to the symmetric variety. The choice of the torus is forced on us by consideration of the real-field case. Let G be a real semisimple Lie group, K a maximal compact subgroup of G, and A a maximal abelian subspace of G/K. Then the Cartan decomposition gives us $K\backslash G/K \cong A/W$, where $W = N_K(A)/Z_K(A)$ is the little Weyl group. Hence it appears that W-equivariant compactifications of A would control compactifications of G/K; this is indeed the case, and the first section of this note gives the definition of analogues of A by Th. Vust. One novelty here is the removal of the char$(k) \neq 2$ restriction commonly seen in the literature.

The second purpose of this note is to give evidence for the existence of a model of canonical compactifications of adjoint symmetric varieties as a smooth scheme over the ring of integers.

We treat here the case of involutions of inner type, and the case of GL_n/O_n.

The construction for the group variety has been carried out by E.Strickland [22] by generalizing the original method of De Concini and Procesi [3] combined with the Cartan decomposition for Chevalley groups over complete valutaion rings. Away from the prime 2, a construction is possible by using again [3] and the ananlogue of Cartan decomposition for symmetric varieties in [26].

Thus the novelty in this case, is the construction of such compactifications for several symmetric varieties, with due regard to the prime 2. The case which encompasses all the difficulties arising in the general case is GL_n/O_n which is treated in detail in §3. It is expected that the canonical compactification of arbitrary adjoint symmetric varieties can be covered by the methods here..

I wish to thank Professor H. Ochiai, the organizer of the conference, for his kind invitation.

2. Compactifications of symmetric varieties.

We first treat the case over an algebraically closed field, and then treat the general (relative) case. The main idea here is that there is a maximal abelian subspace of G/H such that its equivariant closure controls the whole embedding.

It is necessary to recall some basic definitions and properties.

2.1. Notations and definitions for involutions. Let k be an algebraically closed field G be a reductive group scheme defined over k. Contrary to the prevailing custom in the literature on symmetric varieties, we do not assume that the characteristic of k is not equal to 2. We shall see that the main results of the theory holds even for characteristic 2. This development is possible in view of Proposition 2.1.5 where the existence of σ-split tori are proven for non-trivial involutions.

We denote by σ an involution of G defined over k.

Let us recall the following fundamental results of Steinberg. Due to its importance in the sequel, we reproduce its proof.

Theorem 2.1.1. *Let σ be an automorphism of a connected linear reductive group G. Then there exists a Borel subgroup B stable under the action of σ. Moreover, if σ is a semi-simple automorphism of G, then there exists a maximal torus T of B which is stable under σ.*

Proof. The proof proceeds as follows. The σ-twisted action of G on G is defined by $x \to gx\sigma(g)^{-1}$. The first thing to show is that elements of G are twisted conjugate to an element in a σ-stable Borel subgroup. This is shown by showing that the image of the map $\pi : G \times B \to G$ given by $\pi(g,b) = gb\sigma(g)^{-1}$ is closed and contains an open subset of G.

Consider the graph Γ of π: $\Gamma = \{(x,y,b)|y = xb\sigma(x)^{-1}\}$. Γ is a subset of $G\times G\times B$. Let π_i denote the projection to the i-th component. The projection$\widetilde{\Gamma}$ of Γ to $G \times G$ is closed, since it is the inverse image of B under the map $G\times G$ given by $(x,y) \to x^{-1}y\sigma(x)$. Moreover, $\widetilde{\Gamma}$ is stable under right multiplication by $B\times\{1\}$. Hence $S = \Gamma/(B\times\{1\}) \subset G/B \times G$ is a closed subset. G/B is a complete variety, so the second projection of S is closed in G; this projection is equal to the image of π.

To show that π contains an open subset of G, it is enough to show that there is a point on $G \times B$ such that the differential of $\pi : G \times B$ surjects to $\mathfrak{g}$. Let us do the computation at $(1,b)$. Then $d\pi_{(1,b)}(s,t)b^{-1}$ is equal to $(sb+t-bd\sigma(s))b^{-1}$. Since $tb^{-1} \in \mathfrak{b}$, we see that the image of $d\pi_{(1,b)}b^{-1}$ is equal to $\mathfrak{b}+(1-bd\sigma b^{-1})\mathfrak{g}$. Maximal tori in B are conjugate under B-conjugation; hence one can pick b so that $\tau = b\sigma b^{-1}$ fixes a maximal torus T. Since τ permutes the negative roots, it is possible to modify b by an element t of T so that $(1 - t\tau t^{-1})\mathfrak{u}^- = \mathfrak{u}^-$. Now $\mathfrak{g} = \mathfrak{b} + \mathfrak{u}^-$, so $d\pi$ is a surjection at $(1,tb)$.

One then argues as follows. Let B be an arbitrary Borel subgroup of G. Then by conjugacy of Borel subgroups, there exists a $g \in G$ such that $g\sigma(B)g^{-1} = B$. Apply the previous claim to the automorphism $ad(g)\sigma$ with B a stable Borel subgroup. There exists $x \in G$ such that $xbg\sigma(x)^{-1}g^{-1} = g^{-1}$. Then $g = b^{-1}x^{-1}\sigma(x)$. Hence $\sigma(xBx^{-1}) = xBx^{-1}$.

The existence of the stable maximal torus follows by considering the group $< \sigma > \times B$. Since the element σ is semi-simple, there exists a maximal torus S containing σ. $S \cap B = T$ is the desired maximal torus of B.

□

In case the order of σ is prime to the characteristic of k, then σ is a semisimple automorphism.

Definition 2.1.2. An automorphism σ of G is called **quasisemisimple, quass** to be brief, if there exists a Borel subgroup B and a maximal torus $T \subset B$ which are both σ-stable.

Let us note in passing that for any automorphism σ there exists a Borel subgroup B stable under its action. Since maximal tori in B are conjugate to each other under B-conjugation, it follows that there exists a $b \in B$ such that $ad(b)\sigma$ is a quasisemisimple automorphism.

Let us show an example of a non-quass automorphism.

Example 2.1.3. Let $G = SL_2$ and let the characteristic of k be 2. Then $\sigma(X) = {}^tX^{-1}$ is not a quass automorphism. This happens since the fixed point subgroup of σ is isomorphic to G_a.

$$G^\sigma = \{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + b\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}$$

The normalizer B of G^σ is a σ-stable Borel subgroup. We first claim that this is the unique σ-stable Borel subgroup of G. Suppose that P is a σ-stable Borel subgroup of G. Then the unipotent radical of P is given by $[P,P] = R_u(P)$; this is clearly σ-stable. Since $R_u(P) \cong \mathrm{G}_a$ and σ is an involution, the action of σ on the unipotent radical of P is trivial (this is where the characteristic 2 enters). Hence $R_u(P) = G^\sigma$; $P = B$. Suppose further that T is a σ-stable subgroup of B. Since $T \cong \mathrm{G}_m$, there are two possibilities: σ either acts by inversion $x \longrightarrow x^{-1}$ or by the identity map. The first option is ruled out since then σ would turn B into its opposite. The second option is impossible since then we would have $T \subset G^\sigma \cong \mathrm{G}_a$.

On the other hand, the involution τ of G given by $\tau(X) = J^{-1t}X^{-1}J$ where $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a quass automorphism with the set of upper or lower triangular matrices as τ-stable Borels and the set of diagonal matrices as the τ-stable torus.

We then make the following definition relative to σ.

Definition 2.1.4. 1. Let S be a k-split torus. It is called σ-split if and only if the restriction of σ to S is equal to the inverse map: $s \to s^{-1}$.

2. Let P be a parabolic subgroup of G defined over k. P is said to be σ-split if and only if P and P^σ are opposite parabolic subgroups; i.e., if the intersection $P \cap P^\sigma$ is a Levi subgroup of P.
3. Let $A \subset G$ be a k-split torus. We say that A is maximally σ-split if $\dim A$ is the maximum among σ-split tori.

The following is due to Th. Vust for $\mathrm{char}(k) \neq 2$. The sequence of proofs presented here follows his paper closely except for the proof of the existence of σ-split tori for fields of characteristic 2.

Proposition 2.1.5. *Let G be a connected linear reductive group. Let σ be an involution of G. Assume that σ is not the identity map. Then*

1. *There exists a non-trivial σ-split torus.*
2. *Let A be a maximal σ-split torus of G. Then A is the unique maximal σ-split torus of $Z_G(A)$.*
3. *The commutator of $Z_G(A)$ is contained in G^σ: $[Z_G(A), Z_G(A)] \subset G^\sigma$.*
4. $Z_G(A) = (Z_G(A) \cap H)^0 A$.
5. *Let T be a torus of G such that $A \subset T$. Then T is σ-stable.*
6. *Let P be a σ-split minimal k-parabolic subgroup. Then there is a unique maximal σ-split torus A in $L = P \cap P^\sigma$.*
7. *One has $L = Z_G(A)$.*

Proof. 1. Assume that all σ-stable maximal tori are contained in $G^\sigma = H$. We first assume that the characteristic of k is not equal to 2. Let B be an arbitrary Borel subgroup. Then $B \cap \sigma(B)$ is σ-stable. Since σ is semi-simple, there exists a maximal torus T that is σ-stable. By hypothesis, the restriction of σ to T is the identity map; then B is σ-stable. On the other hand, let T be an arbitrary maximal torus of G. There exists a pair of Borel subgroups $B_\pm$ such that $T = B_+ \cap B_-$. Since $\sigma(B_\pm) = B_\pm$, we see that T is σ-stable. But then σ is the identity map on T. Hence σ fixes semisimple elements of G, which is a Zariski dense subset; σ is the identity map.

We now argue the characteristic 2 case. We assume that k is a field of characteristic 2. We first note the following claim.

Claim Suppose that G is semi-simple. Then the group G contains a σ-stable torus.

One can show this as follows. Let G denote the semidirect product of σ with G. Then σ is a unipotent element of G . If the centralizer Z_G (σ) of σ contains a non-central semisimple element, then we are done. Suppose not; then σ is by definition a semi-regular element of G . By going through the table in Spaltenstein's book [20] of semi-regular elements, we are reduced to the following case.

$$\begin{pmatrix}1 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

We now claim that there exists a σ-stable maximal torus.

Let us prove by induction on the semistable rank rank(G) of G. The case where G is a torus, is obvious. By our previous claim, there exists a non-central torus S that is σ-stable. Let $Z = Z_G(S)$ be the centralizer of S in G. Then the semisimple rank of Z is less than rank(G). Hence there exists a σ-stable maximal torus T of Z; it is also a maximal torus of G, and we are done.

Let T be a σ-stable maximal torus. If the action of σ is non-trivial on T, then it is clear that there is a σ-split torus. The case where the action of σ on T is trivial is taken care of by the following claim.

Claim Let k be a field of characteristic 2 and let σ be an involution of G. If there exists a maximal torus T of G such that σ acts by the identity map, then σ is a trivial automorphism.

Let B be a Borel subgroup containing T; it is clear that B is stabilized by σ. Then σ is an inner automorphism, say, $\sigma(x) = gxg^{-1}$. Since σ stabilizes T we see that $g \in N_G(T)$; from the stability of B we see that $g \in B$. Hence $g \in T$; in particular, g is a semisimple element. Since $g^2 \in Z(G)$, $\overline{g} \in G/Z(G)$ is a unipotent element. Therefore $\overline{g} = 1$ in $G/Z(G)$; σ is the identity map.

2. Let S be a σ-split torus of L. It suffices to show that S is contained in A. Clearly, $A.S$ is a σ-split subtorus of $Z_G(A)$. By the maximality of A, $A.S = A$. Hence $S \subset A$.

3. It suffices to show that the restriction of σ to the commutator group $[Z_G(A), Z_G(A)]$ is trivial. Suppose it is not. Then by (1), there exists a non-trivial σ-split torus S. By virtue of (2), S is a subgroup of A, which is a contradiction.

4. Let Z be the center of $Z_G(A)^0$. Then $Z = Z^\sigma A$. Since

$$Z_G(A) = [Z_G(A), Z_G(A)]Z$$

, we see that $Z_G(A) = (Z_G(A) \cap G^\sigma)A$.

5. Let T contain A. Then T centralizes A; hence $T \subset Z_G(A)$. By virtue of 4., for any element t of T, there exists $k \in (Z_G(A) \cap G^\sigma)$ and $s \in A$ such that $t = ks$. Then $\sigma(t)t^{-1} = s^{-2} \in A$. Hence $\sigma(t) = s^{-2}t \in T$. Hence T is σ-stable. □

Let us see how σ-split tori give rise to σ-split parabolics. The following definition is standard.

Proposition 2.1.6. *Let λ be a one-parameter-subgroup (1PS for short) of G. Let $P(\lambda) = \{s \mid \lim_{t\to 0} \lambda(t)s\lambda(t)^{-1}$ exists in $G\}$ and let $L(\lambda)$ be the centralizer of λ in G. Then $P(\lambda)$ is a parabolic subgroup of G, and $L(\lambda)$ is its Levi subgroup.*

Proposition 2.1.7. *Let P be a σ-split parabolic of G. Then by definition, $P \cap \sigma(P)$ is a Levi subgroup; let S be its maximal σ-split torus. Then there exists a 1PS λ of S such that $P = P(\lambda)$ and $L(\lambda) = P \cap \sigma(P)$.*

Proof. We first show the existence of a σ-split λ such that $P(\lambda) = P$. Let T be a maximal torus of L containing S. By 2.1.5, T is σ-stable. The torus T is also a maximal torus of P;hence there exists a 1PS $\mu \in X_*(T)$ such that $P = P(\mu)$. Let F be the face of $X_*(T) \otimes \mathrm{R}$ containing μ. By assumption P is opposite to $\sigma(P)$; hence $-\sigma(\mu) \in F$. Since F is a convex cone, $\lambda = \mu - \sigma(\mu) \in F$. Thus $P = P(\lambda)$ with $\sigma(\lambda) = -\lambda$.

The next step is to show that $\lambda \in X_*(S)$. Since $\lambda \in L$, by maximality of S, we have $\lambda \in X_*(S)$. By virtue of 2.1.6, we see that $P \cap \sigma(P) = Z_G(\lambda) \supset Z_G(S)$. On the other hand, since $S \subset L$, the centralizer $Z_G(S)$ contains a maximal reductive subgroup of P. Hence $P \cap \sigma(P) = Z_G(\lambda) = Z_G(S)$.

□

Corollary 2.1.8. *Let P be a minimal σ-split torus. The following are equivalent.*

1. *P is a minimal σ-split parabolic subgroup of G.*
2. *The Levi subgroup $L = P \cap \sigma(P)$ contains a maximal σ-split torus.*
3. *The Levi subgroup L is the centralizer in G of a σ split maximal torus of G.*

Proof. Let us show that $1 \Rightarrow 2$. Let L denote the σ-stable Levi subgroup of P. Let S be a maximal σ-split torus of L and let $\acute{S}$ be a maximal σ-split torus of G. By virtue of 2.1.7, we know that there exists a 1PS $\lambda \in X_*(S)$ of S such that $P = P(\lambda)$. Identify $X_*(S)$ as a subset of $X_*(\acute{S})$. Let C be a chamber of $X_*(\acute{S}) \otimes \mathrm{R}$ such that $\lambda \in \overline{C}$. If $\acute{\lambda} \in C$, then $P(\acute{\lambda})$ is a σ-split parabolic subgroup of G contained in $P(\lambda)$ with $Z_G(\acute{S})$ as its σ-stable maximal reductive subgroup. Since $P = P(\lambda)$ is minimal, we have $P = P(\acute{\lambda})$. Hence $\acute{S} \subset L$; then $S = \acute{S}$.

The implication $2 \Rightarrow 3$ is a consequence of 2.1.7.

Let us show that $3 \Rightarrow 1$. Let $\acute{P} \subset P$ be a σ-split parabolic subgroup. Let $\acute{S}$ be the maximal σ-split torus of $\acute{P} \cap \sigma(\acute{P})$. Then we have the following.

$$Z_G(\acute{S}) = \acute{P} \cap \sigma(\acute{P}) \subset P \cap \sigma(P) = Z_G(S)$$

Hence $\acute{S} \subset S$. □

Proposition 2.1.9. *Let G be a connected reductive group and let σ be an involution of G. Then any two maximal σ-split tori are conjugate to each other.*

Proof. Let H denote the connected component of the fixed point subgroup G^σ. We first claim that HP is open dense in G. It is enough to show that $\text{Lie}(H) + \text{Lie}(P) = \text{Lie}(G)$, which follows from a standard argument using root subgroups.

Let S and T be maximal σ-split tori of G. Choose 1PS's $\lambda \in X_*(S)$ and $\mu \in X_*(T)$ such that $Z_G(\lambda) = Z_G(S)$ and $Z_G(\mu) = Z_G(T)$. By 2.1.8, we see that the parabolics $P(\lambda)$ and $P(\mu)$ are both σ-split minimal parabolics of G. By our previous claim, there exists an element g of H such that $gP(\lambda)g^{-1} = P(\mu)$. It is clear that

$$g(P(\lambda) \cap \sigma(P(\lambda)))g^{-1} = P(\mu) \cap \sigma(P(\mu))$$

Hence $g(Z_G(S))g^{-1} = Z_G(T)$; but then S and T are the unique maximal σ-split tori inside their respective centralizers $Z_G(S)$ and $Z_G(T)$. Therefore we have $gSg^{-1} = T$.

□

We fix some subvarieties of G to work with. Define the map q by setting

$$q(g) = \sigma(g)g^{-1}$$

and set P equal to the image of q: this is a subvariety of G.

Then it is known by Richardson [15] that

Proposition 2.1.10. *Let the characteristic of k not equal to 2. Then the map $q : G \to P$ descends to a biregular map $\tilde{q} : G/H \to P$.*

Proof. This follows from computation of the differential of $\tilde{q}$.

□

Remark 2.1.11. In general, this map is inseparable for $\text{char}(k) \neq 2$.

Definition 2.1.12. Denote by Q the subvariety of σ^{-1}-fixed points in G.

We also assemble some facts concerning the relative root system with respect to σ. First we assemble some generalities on root systems for arbitrary σ-stable maximal tori T. Let $R = R(G,T)$ be the root system with respect to T. On $X = X^*(T)$ there is an induced action of $\sigma : \chi \to \chi(\sigma)$. Since σ acts on G, $\sigma(R) = R$. The following definitions are due to Vogan.

Definition 2.1.13. A root $\alpha \in R$ is

1. **real** if $\sigma(\alpha) = -\alpha$.
2. **complex** if $\sigma(\alpha) \neq \pm\alpha$.
3. **compact imaginary** if $\sigma(\alpha) = \alpha$ and $\sigma(X_\alpha) = X_\alpha$.
4. **non-compact imaginary** if $\sigma(\alpha) = \alpha$ and $\sigma(X_\alpha) = -X_\alpha$.

The following propositions are well-known.

Proposition 2.1.14. *Let A be a maximal σ-split torus. Let $A \subset T$ be a maximal torus containing A. Then T is σ-stable. Let $R = R(G,T)$. If $\sigma(\alpha) = \alpha$ for $\alpha \in R$, then α is a compact imaginary root.*

Proposition 2.1.15. *The set of compact imaginary roots together with T^σ is the centralizer of A in G.*

Let A be a maximal σ-split torus of G. Let T be a maximal torus of G containing A. T is automatically σ-stable. Let $R = R(T,G)$ be the root system of G with respect to T. Consider the projection map $\rho : X^*(T) \to X^*(A)$. The kernel of ρ is denoted by R_0, and the complement of R_0 in R by R_1.

The image of R_1 in X is called the restricted root system of the pair $(G, \sigma$ in view of the following proposition, originally due to Richardson [15].

Proposition 2.1.16. *The image of R_1 in X forms a (non)-reduced root system in $X^*(A) \times \mathrm{R}$. The Weyl group of this system is called the little Weyl group of the pair $(G, \sigma$.*

2.2. **Compactifications over algebraically closed fields.** Let k be an algebraically closed field. Let (G, σ) be a reductive symmetric pair. We fix a choice of a maximal σ-split torus A of G. Let W denote the little Weyl group 2.1.16.

Definition 2.2.1. Let X be a normal G-variety, and let $\iota : G/H \to X$ be an open immersion. The pair (X, ι) is called a G-embedding of G/H.

Definition 2.2.2. Let A be a maximal σ-split torus of G. Let A_X be the closure of A in X.

The following has been conjectured in [26].

Theorem 2.2.3. *The functor $X \to X_A$ gives a fully faithful functor from the category of G-embeddings of G/H to the category of W-stable torus embeddings of A.*

The basis for this theorem is the following analogue of the Cartan decomposition ([26]).

Theorem 2.2.4. *Let k be an algebraically closed field. Let G be a reductive group over k, and let σ be an involution of G defined over k. Let A be a maximal σ-split torus of G and let W be the little Weyl group. By $k((t))$ and $k[[t]]$, we denote the field of Laurent power series in the variable t and the ring of formal power series in t. We fix an embedding of the group $X_*(A)$ of one parameter subgroups of A into $G_{k((t))}$.*

Then we have the following decomposition.

$$G_{k[[t]]} \backslash (G/H)_{k((t))} \cong X_*(A)/W$$

We now explain how this classification relates to the theory of coloured fans of Luna and Vust. We first give a brief description of their theory as generalized by Brion, Pauer, and Knop.

The basic data for their classification is:

1. G-invariant valuations of $k(G/H)$, denoted by $\mathcal{V}$.
2. B-stable divisors on G/H, denoted by $\mathfrak{D}$

Our starting point is the following identification of G-invariant valuations. This is an easy consequence of Luna-Vust [10]. First recall that giving a valuation v of $k(G/H)$ is equivalent to giving a homomorphism $k(G/H) \to k((t))$ up to twisting by $\mathrm{Aut}_k k[[t]]$. A valuation v is G-invariant if and only if the associated map $k(G/H) \to k((t))$ is equivalent to itself after moving by elements of $G(k[[t]])$. Since homomorphisms from $k(G/H)$ to $k((t))$ are simply $k((t))$-rational points of G/H, we see that we have the following identification.

Proposition 2.2.5. *The set of G-invariant valuations of $k(G/H)$ is in one-to-one correspondence with $X_*(A)/W$.*

It is now necessary to classify B-eigenfunctions in $k(G/H)$. We fix a Borel subgroup B of G such that BH is dense in G. There exists a σ-stable maximal torus T of B. Let $B = TU$ be the Levi decomposition of B with U the unipotent radical of B.

Proposition 2.2.6. *Let χ be a character of T, extended to a character of B. Then a non-zero χ-eigenfunction of B in $k(G/H)$ exists if and only if the restriction of χ to $T \cap H$ is trivial.*

Proof. Let f be a non-zero χ-eigenfunction of B in $k(G/H)$. Then there exists an open set U in G/H such that f is regular on U. Since BH is open in G, we have $U \cap BH \neq \emptyset$. By B-equivariance, f is defined and non-zero on BH. Let $t \in T \cap H$. Then $f(H/H) = f(tH/H) = \chi(t)f(H/H)$. Hence $\chi(t) = 1$.

Let us show the converse. We construct a non-zero χ-eigenfunction of B. It will be an element of $Ind_B^G\chi$. Define f on BH/H by the following formula.

$$f(H/H) = 1$$

$$f(tuH) = \chi(t), u \in U, t \in T$$

f is well-defined and regular on BH/H in view of the following decomposition.

$$BH/H \cong B/(B \cap H) \cong T/(T \cap H) \times U/(U \cap H)$$

□

Proposition 2.2.7. *The map $T \to G/H \cong S$ defined by $t \to t\sigma(t)^{-1}$ induces an isomorphism*

$$T/(T \cap H) \cong A \to S$$

Hence via this identification, we have $X^*(T/T \cap H) \cong X^*(A)$.

Several examples are in order:

Example 2.2.8. Let $G = H \times H$ and S a maximal torus of H. Then $T = S \times S$, and $T \cap \Delta(H) = \Delta S$ is connected.

Example 2.2.9. $G = SL_n$, $\sigma(X) = {}^tX^{-1}$. Then T is the set of diagonal matrices, and $T \cap H \cong (Z/2Z)^{n-1}$. Hence $T \cap H$ is not connected.

Example 2.2.10. Let $G = SL_{2n}$ and $\sigma(X) = J^tX^{-1}J^{-1}$. Then T is the set of diagonal matrices. $T \cap H$ is the maximal torus of $H \cong Sp_n$, hence connected.

We now turn to the identification of B-stable divisors on G/H. These are actually given by the divisors associated to B-eigenfunctions in $k(G/H)$. Thus we have the following identification.

Proposition 2.2.11. *The set of B-stable effective divisors on G/H are identified with codimension 1 faces of the Weyl chamber in $X_*(A)$.*

Thus we have the following method for giving coloured fans from our description. Let A_X be the torus embedding associated to X. Denote by $Fan(X)$ the corresponding fan. Simple embeddings are in one-to-one correspondence with W-orbits of faces in $Fan(X)$. Take one face F, and move it by the action of W so that wF has non-empty intersection with the dominant Weyl chamber C. Then call $\mathcal{V}_F$ the set of edges of $wF \cap C$ belonging to F and call $\mathcal{D}_F$ the set of edges of $wF \cap C$ which do not belong to F. Then we have the following proposition.

Proposition 2.2.12. *The collection of coloured fans $\{(V_F, D_F)\}$ for F running through representatives of $Fan(X)/W$ gives the coloured fan associated to X.*

2.3. **Embeddings over arbitrary fields.** Our next task is to see what happens for non-algebraically closed fields. Let us recall the theory of torus embeddings for arbitrary fields.

Definition 2.3.1. A torus T over k is a linear algebraic group, which after base extension to $\bar{k}$ becomes the product of copies of G_m, the multiplicative group. The number of copies is called the rank of the torus.

It is a theorem of Grothendieck that T splits after a separable field extension. Let k_s denote the separable closure of k. Then we have the following correspondence.

Proposition 2.3.2. *Let $\Gamma = Gal(k_s/k)$. Then there is a one-to-one correspondence between tori defined over k of rank ℓ and $\mathbb{Z}[\Gamma]$-module structures on $\mathbb{Z}^{\ell}$.*

Thus, given a torus defined over k, we have the action of the Galois group Γ on $X_*(T)$ and dually on $X^*(T)$. We have the following proposition for torus embedding defined over k.

Proposition 2.3.3. *A fan F corresponds to a torus embedding defined over k if and only if it is stable under the action of Γ on $X_*(T)$.*

This easily generalizes to our case. Let A be a maximal σ-split torus of G defined over k which contains a maximal $(k, \sigma$-split torus of G.

Proposition 2.3.4. *A fan for A gives an embedding for G/H if and only if it is stable under the action of Γ.*

In order to prove this, we need to recall some facts from Galois cohomology [19]. The action of an element s of Γ produces a twist of X, which is isomorphic to X over K by our assumption. Denote by c_s this isomorphism. By the uniqueness of isomorphisms, we see that c_s is a 1-cocycle of Γ. Thus it is possible to restrict the field of definition of X by taking the quotient $\Pi_{s\in\Gamma}X/\Gamma$.

2.4. **Compactifications arising from representations.** One way of compactifying symmetric varieties of adjoint type is by means of representations. Let V be a rational G-module; we suppose that V contains a nonzero H-eigenvector v. Then the closure of the map $G/H \to \mathrm{P}(V)$ given by $gH \to gv \in \mathrm{P}(V)$ gives a compactification of G/H. The maximal σ-split torus is again denoted by A.

Let us first investigate what happens for the torus case; the answer is in the form of combinatorics of weights appearing in the weight vector decomposition of v.

Example 2.4.1. Let us consider the case of torus embeddings. There are two ways of constructing torus embeddings. The first is to let T act on a complete variety X and take the closure of the orbit of a point x; the second is to glue torus embeddings together.

In this example, we treat the first method in detail. We first consider the action of T on a projective space. Let V be a rational T-representation. Let $v \in V$ be a non-zero vector. Denote by $v = \sum_{\chi\in X^*(T)} v_\chi$ its decomposition into T-weight vectors; $\rho(t)v_\chi = \chi(t)v_\chi$. The support $\mathrm{supp}(v)$ of v is by definition the set of χ for which $v_\chi \neq 0$. Then, for $\overline{Tv} \subset \mathrm{P}(V)$ to become a torus embedding, it is necessary and sufficient that the $\mathbb{Z}$-span of $\mathrm{supp}(v)$ equal $X^*(T)$.

The fan corresponding to $T \subset \overline{Tv}$ is given by the dual cone of the convex hull of $\mathrm{supp}(v)$.

This construction can be generalized as follows. Let Y be a complete variety with T-action, equipped with a T-linearized line bundle L. Let

X be the closure of a T-orbit of a point x in Y. The fan of X can be determined as follows. Let X^T be the set of T-fixed points in X; this is a finite set. Consider the T-equivariant inclusion $X^T \to X$. i^*L is a T-linearized line bundle on X^T; hence it determines a set of characters of T. Call this set $\mathrm{supp}(x, L)$, the support of x. The fan associated to X is the dual cone of the convex hull of $\mathrm{supp}(x, L)$.

Proposition 2.4.2. *Let V be a rational G-module with a nonzero H-fixed vector v. Then the closure $\overline{Gv} \subset \mathrm{P}(V)$ is normal if and only if $\overline{Av} \subset \mathrm{P}(V)$ is normal.*

The G-orbit closures of H-fixed vectors for irreducible G-modules are important for the theory of compactifications. They were first studied by Satake [17] in the real case.

Proposition 2.4.3. *Let V_λ be an irreducible rational G-module with a nonzero H-fixed vector v. Then the closure $\overline{Gv} \subset \mathrm{P}(V)$, which we denote by $X(\lambda)$, only depends on the support* $\mathrm{supp}(\lambda)$ *of λ.*

The following proposition answers question posed by G.Heckman.

Proposition 2.4.4. *The necessary and sufficient condition for $\overline{G.v} \in \mathrm{P}(V_\lambda)$ to be non-singular is that $\mathcal{D} - \mathrm{supp}(\lambda)$ to be connected and of type A.*

We consider an empty set as a connected diagram of type A. If the highest weight λ is regular in the sense that $\mathcal{D} = \mathrm{supp}(\lambda)$, then the compactification $X(\lambda)$ is automatically smooth. This is the canonical compactification.

Example 2.4.5. This proposition implies that the unique equivariant compactifications of rank 1 symmetric varieties are non-singular. In the real case, these symmetric varieties are associated to hyperbolic geometries over the reals R, the complex numbers C, the quarternions H, and the octonians 0. There is a canonical way to construct these compactifications. The homogeneous spaces are given by $SL(n+1)/GL(n)$, $SO(n+1)/SO(n)$, $Sp(n+1)/Sp(n) \times Sp(1)$ and $F_4/Spin(9)$.

Let us first consider the classical cases.

Case (1). Let V be the standard representation of SL_{n+1}. Let V^* denote the dual of V; $V^* = \mathrm{Hom}_k(V, k)$. Consider $X = \mathrm{P}(V) \times \mathrm{P}(V^*)$. There are two orbits of G on X;$\{(v, \lambda) | \lambda(v) \neq 0\}$ and $\{(v, \lambda) | \lambda(v) = 0\}$. Note that both conditions are given as incidence conditions between the point v and the hyperplane determined by $\lambda = 0$. A pair (v, λ) in the open orbit determines a splitting of $V = V_1 \oplus V_2$ where $V_1 = kv$, $V_2 = \mathrm{Ker}\lambda$. Hence the stabilizer of (v, λ) is $GL_n \subset SL_{n+1}$.

Case (2). Let V be the standard representation of SO_{n+1}. Let $<,>$ be the defining symmetric bilinear form on V. Hence via this form, we have an SO_{n+1}-isomorphism $V \cong V^*$. Consider $G/P \times G/P \subset \mathrm{P}(V) \times \mathrm{P}(V^*)$, where G/P is the space of isotropic vectors. This is the compactification of the rank 1 symmetric variety $SO(n+1)/SO(n) \times SO(1)$.

Case (3). The compactification of $Sp(n+1)/Sp(n) \times Sp(1)$ can be dealt with in a similar way. Let $V = W \oplus W^*$ be the decomposition of the underlying vector space with respect to a maximal isotropic subspace W. Fix a decomposition of $W = W_1 \oplus W_2$, where $\dim W_1 = 1$ and $\dim W_2 = n$. Let P_i be the stabilizer of W_i. Then the compactification is $G/P_1 \times G/P_2$ with the diagonal G action.

Case (4). This is the case F_4/B_4. A model can be constructed as follows. Let 0 be the Cayley algebra over k. It is well known that it is unique up to isomorphisms over an algebraically closed field. Consider J, the Jordan algebra of 3×3 Hermitian matrices over 0. The Cayley projective plane $\mathrm{P}^2(0)$ is by definition the space of primitive idempotents in J. The open orbit consists of $x \in J$ such that $(x,x) \neq 0$, and the closed orbit of $x \in J$ such that $(x,x) = 0$. $\mathrm{Aut}(J) \cong J$, and the stabilizer of x is $Spin_9$.

Example 2.4.6. The previous example generalizes as follows. We can construct the canonical compactification of $GL(n)/GL(p) \times GL(q)$ after a sequence of blow-ups. Let $X = G/P_p \times G/P_q$. Consider the diagonal action of G on X. The G-orbits of X are given by $X_r = \{(V_1, V_2) | \dim(V_1 \cap V_2) = r\}$. The closure relation is given by $\overline{X_r} \subset X_s$ if and only if $r \geq s$. The singular locus of X_r is X_{r-1}. The canonical compactification is given by successively blowing up the orbits X_p, X_{p-1}, ... The open orbit is clearly isomorphic to $GL(n)/GL(p) \times GL(q)$.

Example 2.4.7. Let us consider the construction of Piatetski-Shapiro and Rallis [14]. In their case, a compactification of classical group G is constructed as follows. Consider a homomorphism $G \times G \to L$ such that there exists a parabolic P of L such that $G \times G \cap P = \Delta G$. Then $G \times G \to L/P$ induces a $G \times G$-equivariant compactification $G \to L/P$. The homomorphism is produced as follows. Let V be a vector space over k. Let $(,)$ be a non-degenerate bilinear form, which is either alternating or symmetric. Then G is the stabilizer of $(,)$. Let $W = V \oplus V$. Define a bilinear form $<,>$, by doubling the variables, as follows. $<(v_1, v_2), (w_1, w_2)> = (v_1, w_1) - (v_2, w_2)$. L is given as the stabilizer of $<,>$. ΔV is a maximal isotropic subspace of W. Let P be the stabilizer of ΔV. There is an obvious embedding of $G \times G$ into L. It turns out that this homomorphism satisfies the condition above.

Let us identify this compactification. Set $\dim V = n$. Let T be a maximal torus of G. Note that for graphs of type B, C and D there is a unique vertex to delete to produce a diagram of type A. The representation constructed above corresponds to this vertex.

The case of the unitary group can also be treated in a similar way.

2.5. **Compactifications arising from commuting involutions.** Let σ and τ be involutions of G, defined over k, such that $\sigma\tau = \tau\sigma$. Such pairs have been classified by Berger, Oshima-Sekiguchi [13], Helminck [5, 6]. For a study of the double coset decomposition the reader is referred to recent work of T. Matsuki [11].

Such pairs arise as follows. Let $G_{\mathbb{R}}$ be a real semisimple linear group and let $G_{\mathbb{C}}$ be its complexification. Let σ be the an antiholomorphic map of $G_{\mathbb{C}}$ given as complex conjugation with respect to $G_{\mathbb{R}}$. Then there exists a Cartan involution τ of $G_{\mathbb{C}}$ such that $\sigma\tau = \tau\sigma$. The involution τ is unique up to conjugation with respect to $\mathrm{Ad}G_{\mathbb{R}}$.

A pair of commuting involutions σ and τ of G give rise to pairs of symmetric varieties

$$G^\sigma/(G^\sigma \cap G^\tau) \to G/G^\tau$$

and

$$G^\tau/(G^\sigma \cap G^\tau) \to G/G^\sigma.$$

An equivariant embedding $G/G^\tau \to X$ gives rise to an embedding

$$G^\sigma/(G^\tau \cap G^\sigma) \to X$$

by taking the closure of the image in X. The purpose of this section is to identify this functor in terms of the split torus picture. In particular, we see that the canonical compactification of G/G^τ goes to the canonical compactification of $G^\sigma/(G^\tau \cap G^\sigma)$.

Definition 2.5.1. Let σ and τ be a pair of commuting involutions of G. Let A_0 be a maximal σ- and τ-split torus. Let A_1 be a maximal σ-split torus containing A_0, and let A_2 be a maximal τ-split torus. Let W_i denote the corresponding little Weyl groups of A_i for $i \in \{0, 1, 2\}$.

Proposition 2.5.2. *Let σ and τ be a pair of commuting involutions of G. Consider the following embedding.*

$$G^\tau/(G^\sigma \cap G^\tau) \to G/G^\sigma.$$

The correspondence between G/G^σ-embeddings to $[G^\tau/(G^\sigma \cap G^\tau)$-embeddings is given as follows.

Let X_1 be a G/G^σ-embedding. Let $\mathfrak{F}$ be the fan associated to X_1. Then the fan associated to X_0, the closure of $G^\tau/(G^\sigma \cap G^\tau)$ inside X_1 is given by taking for each face F of $\mathfrak{F}$, the face determined by non-empty intersection $F \cap X_*(A_0)$.

By using this proposition, one can show that canonical compactifications correspond to canonical compactifications.

Proposition 2.5.3. *Let X_1 be the canonical compactification of G/G^σ. Then the closure of $G^\tau/(G^\sigma \cap G^\tau)$ in X_1 is the canonical compactification of $G^\tau/(G^\sigma \cap G^\tau)$.*

3. Canonical compactifications over the ring of integers

In this section, we give a construction of canonical compactifications over the ring of integers. We assume that G is split over k and σ is an involution of G Let k be an algebraically closed field k. Let σ be an involution of G defined over k. We first show that it is possible to associate to such data a model over $\mathbb{Z}$, the ring of integers. This model is split in the sense that

1. G is split over $\mathbb{Z}$
2. There exists a σ-split maximal subtorus A of G which is split over $\mathbb{Z}$

It is for such a pair that we construct our canonical model.

3.1. The case of inner involutions. Recall that an automorphism of G is inner if it is of the form $\sigma(x) = gxg^{-1}$ for some element g of G. If the rank of G is equal to G^σ, then σ is an inner automorphism. The converse also holds. Symmetric varieties arising from inner involutions are called symmetric varieties of inner type. The fixed point subgroup G^σ is the Levi subgroup of a maximal parabolic subgroup P of G. Hence G/H embeds as the open orbit of $G/P \times G/P^{op}$, where P^{op} denotes the parabolic subgroup opposite to P, making it possible to generalize the method of example 2.4.6.

Proposition 3.1.1. *Let G be a semisimple group of adjoint type. Let σ be an inner involution, and let P be a maximal parabolic subgroup such that $G^\sigma = P \cap P^{op}$. Let T be a maximal torus of $P \cap P^{op}$, let G be the Chevalley group scheme with respect to this choice of maximal torus. Consider the scheme $X = \mathrm{G}/\mathrm{P} \times \mathrm{G}/\mathrm{P}^{op}$ over $\mathbb{Z}$. Then the successive blow-up of orbits of X gives a model over $\mathbb{Z}$ of the canonical compactification of G/H.*

4. An example: the theory of complete conics over the ring of integers

This construction served as motivation for our theory. What we tried to explain is the following curious fact.

1. The number of conics tangent to 5 given conics is equal to 3264 if the characteristic of k is not equal to 2.
2. The number of conics tangent to 5 given conics is equal to 51 if the characteristic of k is equal to 2

The first is (in essence) results of Schubert and DeConcini-Procesi. The second is due to Vainsencher [27]. A curiousity is the equality $51 = 3264/2^6$. We will find a natural explanation of this equality by using this compactification, and by noting that several divisors attain multiplicity when specialized to 2.

Let us recall in geometric language the results of Schubert, DeConcini-Procesi, and Vainsencher.

A naive approach to the problem consisted in the following: to compactify the space of conics to a projective space of dimension 5. Then the divisor of conics tangent to a given conic is a divisor of degree 6. Hence a naive count of the intersection number of 5 such divisors is 6^5. This approach does not give the correct number precisely since the locus of double lines, which appear as the closed orbit, is contained in all the divisors arising. This comes from the fact that double lines intersect each conic with multiplicity 2, thus being recognized as tangent to the conic. A way to overcome this difficulty has been devised by Chasles. It simply consists in blowing up the loci of double lines. In geometric language of the time, this amounts to taking the degeneration of the dual curve into account.

To recapitulate, we have three ways of viewing this compactification. (Called complete conics by Schubert).

1. X is the blow-up of P^5 centered at the loci of double lines.
2. For a non-singular conic C, let $\check{C}$ denote the dual curve of C. Consider the closure of $\{(C, \check{C})\}$ in $P^5 \times \check{P}^5$.
3. Consider a regular dominant representation of GL_3 with H-fixed vector. Let v be the unique (up to scalar) H-fixed vector. Consider the closure of $G.v$ in $\mathrm{P}(V)$

It can be shown that all three constructions give isomorphic embeddings of G/H. (Actually, it is a simple exercise using results from §1).

An apparent difficulty in the case of characteristic 2, is the existence of the "strange" points. Let us note the following general fact. Consider a non-singular plane curve C. Denote by $\check{C}$ the dual curve of C. It is known that C and $\check{C}$ are birationally equivalent if and only if $deg(C) = 2$. There is a natural map $d : C \to \check{C}$ which associates to a point p of C the tangent line $\ell = T_pC$ passing through p. It is know that d is a birational map if and only if C is of degree 2 and the characteristic of k is not equal to 2. What happens in the case of $deg(C) = 2$ and $char(k) = 2$ is that $\check{C} \equiv \mathrm{P}^1$, and the degree of the map d is 2. In fact, d is an inseparable map. Hence in this case, the dual curve $\check{C}$ is the pencil of lines passing through a point. This point $st(C)$ is called the strange point of C. Vainsencher's idea consisted in showing that the closure of the correspondence $(C, st(C))$ gives a regular compactification of the space of conics with divisors with normal crossings.

4.1. **The construction.** In characteristic $p \neq 2$, quadratic forms $\sigma b_{ij} x_i x_j$ correspond to symmetric matrices $A = (a_{ij})$ via the correspondence

$$a_{ij} = \frac{b_{ij} + b_{ji}}{2}$$

A quadratic form in n-variables defines a quadratic hypersurface in the projective space P^{n-1} of $n-1$ dimension. The tangent planes to this hypersurface forms yet another hypersurface in the dual projective space $\check{P}^{n-1}$. The equation of the polar hypersurface (i.e., the loci of tangent planes to the hypersurface) is given by the symmetric matrix of $(n-1) \times (n-1)$ minors of A.

In characteristic 2, the correspondence obviously breaks down. We rectify the situation as follows. Consider the space of symmetric matrices $Symm$ as an affine scheme over $\mathbb{Z}$. Let $\pi : Symm \longrightarrow Symm$ be defined by $\pi(x_{ij}) = x_{ij}$ if $i \neq j$, and $\pi(x_{ii}) = 2x_{ii}$. Localized at primes not equal to 2, π gives an isomorphism of schemes. The diagonal disappears at the fiber of 2. This should be viewed as the blow-up of the scheme of symmetric matrices along the ideal $(2x_{ii})$. This is consistent with the traditional treatment of quadratic forms, where one associates to a quadratic form Q the alternating bilinear form $f(x,y) = Q(x) + Q(y) + Q(x+y)$.

Proposition 4.1.1. *The action $A \longrightarrow XAX'$ on $Symm$ lifts to π : $Symm \longrightarrow Symm$.*

We need to perform blow-ups along ideals generated by $k \times k$ minors. Let $A = (a_{ij})$ be a symmetric matrix.

Proposition 4.1.2. *Let $P(x) = \det(\pi(A))$. Then*

1. *If n is even, then there exists a polynomial $Pf(A)$ in x such that $P(x) \cong Pf(A)^2 \pmod 2$.*
2. *If n is odd, then $P(x)/2$ is an integer coefficient polynomial, which after reduction modulo 2 is non zero. We denote by $\mathrm{rdet}(A)$ the integer coefficient polynomial $P(x)/2$.*

We shall now turn to the specific case at hand. Let $n = 3$. Then an easy calculation shows:

Proposition 4.1.3. *Let X be the scheme over $\mathbb{Z}$ defined by localizing $\mathbb{P}Symm$ with respect to the ideal $(\mathrm{rdet}(A))$. Let I_2 be the ideal generated by the 2×2-minors of $\pi(A)$. Let $\tilde{X}$ be the blow up of $\mathbb{P}Symm$ with respect to I_2. Then*

1. *Let k be an algebraically closed field. Then $X \otimes k$ is isomorphic to PGL_3/O_3.*
2. *$\tilde{X}$ is a regular scheme over $\mathbb{Z}$.*
3. *$\tilde{X} \times k$ is isomorphic to the canonical compactification for algebraically closed fields k of characteristic not equal to 2.*
4. *$\tilde{X} \times k$ is isomorphic to the Vainsencher compactification for algebraically closed fields k of characteristic 2.*

Proof. The key to the proof (besides some explicit computations) is to note that the strange point of the conic associated to A is given by (a_{12}, a_{13}, a_{23}) in coordinates and that I_2 reduced $mod2$ is equal to $(a_{12}, a_{13}, a_{23})^2$. □

We now turn to an explanation of the equality $3264/2^5 = 51$. The method of computation , originally due to Chasles, is as follows.

We first fix a flag $p \in \ell$, where ℓ is a line and p is a point on ℓ. Let B_1 denote the divisor of conics passing through p, and let B_2 denote the divisor of conics tangent to the line ℓ. The flag $p \in \ell$ determines a Borel subgroup: let B denote the stabilizer of $p \in \ell$. Then it is clear that B_1 and B_2 are the Borel stable divisors of G/H. Let D_1 be the divisor of double lines with two pencils as tangency conditions. Let D_2 denote the divisor of two lines. Finally, let D denote the divisor of conics tangent to a given conic. Then in the Chow ring of X, we have the identity $D = 2B_1 + 2B_2$. The computation is completed using a condition between the degeneracy divisors and the incidence divisors. For details, consult the book of Schubert [18].

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Mathematical Institute, Tohoku University
Current address: Department of Mathematics, MIT, 77 Mass. Ave., Cambridge, MA 02139
E-mail address: `uzawa@math.tohoku.ac.jp`